Machine-Checked Spectral Gaps and Multiplet Theorems via the Finite-Temperature Purity Bootstrap in Lean 4

Dale Richards-Smith
RS9 Systems (rs9.com.au)
lab@rs9.com.au

November 26, 2026

Abstract

We formalize the finite-temperature purity bootstrap for quantum spin systems in Lean 4 (`v4.36.0`, Mathlib `v4.36.0`) across 25 modules and 1,509 verified theorem constants (1,288 under `PurityBootstrap.*`) with zero `sorry`, zero `admit`, zero `native_decide`, and zero custom axioms—strictly within standard axioms `[propext,Classical.choice,Quot.sound]`. We establish three main results:

1. **Unconditional 2D Coupling-Rectangle Multiplet Theorem (Candidate E):** Inside Lean 4's pure kernel, with zero external numerical inputs, we prove that for every coupling pair $(g,h) \in [\frac{15}{128}, \frac{16}{128}] \times [\frac{47}{128}, \frac{48}{128}]$ and every even ring length $\ell = 2m \geq 64$, the interacting transverse-field Ising ring $H_\ell(g,h)$ exhibits an exponentially narrow ground doublet $0 \leq E_1(\ell) - E_0(\ell) \leq \frac{1}{20} e^{-(\log 20)\ell/128}$ separated by a uniform spectral gap $E_i(\ell) - E_1(\ell) > \frac{\log 8}{48} \approx 0.0433217$ for all $2 \leq i < 2^\ell$.
2. **Zero-Abstract-Transfer Spin-1 Haldane Gap at $(\Delta, D) = (1, 1/4)$ (Candidate A):** Using a finite-M no-collision $O(2)/SO(3)$-equivariant real symmetric transfer matrix, exact cycle contraction, first-order Trotter convergence, subsequential convergence of signed moments under a uniform Hilbert–Schmidt ℓ^2 bound, and real $D_4 + R_z(\pi/4)$ sector projections inside Lean 4, we eliminate *all* abstract transfer-operator and signed-sector-moment hypotheses. Taking *only* 88 finite physical thermal trace enclosures ($n = 4 \ldots 14$, $b \in \{31/2, 23\}$, seams $\{I, R_z(\pi), R_x(\pi), R_z(\pi/2)\}$) and a single $L = 112$ real variational Rayleigh witness, Lean 4 proves a unique ground state (multiplicity 1 in `charpoly.roots`) and uniform spectral gap $\gamma_L > \frac{\log 20}{368} \approx 0.00814058$ on the concrete $3^L \times 3^L$ physical spin-1 Hamiltonian $H_{\text{spin1}}(L, 1/4, I)$ for every finite even $L \geq 896$. All 88 thermal enclosures are certified from archived 256-bit integer totals in exact rational arithmetic (worst endpoint distance $1.245 \times 10^{-6} < 10^{-5}$, 87.55% safety margin), alongside an exact `int64`-residual 7-site open-chain lower bound and a 160-bit Arb enclosure of a $\chi = 32$ periodic MPS trial witness at $L = 112$.
3. **Zero-Abstract-Transfer Isotropic Spin-1 Haldane Gap at $(\Delta, D) = (1, 0)$ (OpenAI Source 1):** Applying the same concrete transfer and $D_2 \rtimes C_3$ projection bridge inside Lean 4 yields a unique ground state and uniform gap $\gamma_L > \frac{\log 20}{784}$ on $H_{\text{spin1}}(L, 0, I)$ for every finite even $L \geq 2304$, taking only the 18 concrete physical thermal trace enclosures ($n = 4 \ldots 12$, $b = 49/4$, seams $\{I, R_x(\pi)\}$) and a single $L = 72$ variational Rayleigh witness.

1 Introduction

Proving uniform spectral gaps for infinite families of interacting quantum many-body Hamiltonians is difficult because the Hilbert-space dimension grows exponentially with system size (e.g.,

Model	Lean Declaration	Scope / Uniformity	External Inputs	Proven Spectral Bound
Candidate E (1D TFIM, 2D coupling box)	`candidate_e_unconditional_full_hamiltonian_theorem`	All $(g,h) \in [\frac{15}{128}, \frac{16}{128}] \times [\frac{47}{128}, \frac{48}{128}]$, finite even $\ell \geq 64$	**None** (100% unconditional in Lean 4)	$0 \leq E_1 - E_0 \leq \frac{1}{20} e^{-\frac{\log 20}{128}\ell}$, $E_i - E_1 > \frac{\log 8}{48}$ $(i \geq 2)$
Candidate A (Spin-1, $\Delta = 1, D = \frac{1}{4}$)	`candidate_a_physical_spin1_charpoly_gap_896`	Concrete $3^L \times 3^L$ H_{spin1}, all finite even $L \geq 896$	88 finite Z_{spin1} boxes ($n \leq 14$) + 1 trial ψ ($L = 112$)	Unique ground state; $\gamma_L > \frac{\log 20}{368} \approx 0.00814058$
Source 1 (Spin-1, $\Delta = 1, D = 0$)	`source1_physical_spin1_charpoly_gap_2304`	Concrete $3^L \times 3^L$ H_{spin1}, all finite even $L \geq 2304$	18 finite Z_{spin1} boxes ($n \leq 12$) + 1 trial ψ ($L = 72$)	Unique ground state; $\gamma_L > \frac{\log 20}{784}$

Table 1: Summary of the three flagship Lean 4 theorems in `haldane-purity-lean`. All abstract transfer-operator (`SpectralSystem1D`) and signed-sector-moment hypotheses have been eliminated inside Lean 4 for all three models.

3^L basis configurations for a spin-1 chain of length L, exceeding 10^{427} at $L = 896$). The *finite-temperature purity bootstrap*, introduced in the September 24, 2026 periodic spin-1 Haldane gap preprint [1] and extended to symmetry-twisted sectors and higher spatial dimensions by Grover [2], overcomes this exponential barrier by turning finite-volume thermal trace enclosures and a single variational energy witness into coupled spatial- and temporal-doubling purity contractions that hold uniformly for all sufficiently large finite even rings.

While the purity bootstrap reduces an infinite family of spectral problems to finite arithmetic checks, its end-to-end rigor rests on two delicate layers: (i) the algebraic, analytic, and representation-theoretic transfer-operator machinery linking physical Gibbs traces $\mathrm{Tr}(e^{-b(H_n + anI)})$ to signed sector-moment expansions across both even and odd ring lengths, and (ii) the exact interval arithmetic and coupled bootstrap inequalities themselves. In this work, we formalize *both* layers inside Lean 4's pure kernel in the repository `haldane-purity-lean`.

Honest verification boundary. Table 1 states the exact scope of each result. Candidate E is *completely unconditional* inside Lean 4's kernel: no external numerical oracle is used anywhere. For Candidate A and OpenAI Source 1, *every* abstract mathematical step—including the concrete $3^L \times 3^L$ physical spin-1 Hamiltonian, explicit single-site unitary equivalence to the textbook S^z weight basis (yielding many-body equivalence via tensoring), finite-M transfer matrix, Trotter convergence, subsequential signed-moment convergence under a uniform ℓ^2 bound, exact D_4 and D_2 symmetry projections, nonnegative polynomial filters (sixteen degree-14 filters for Candidate A; degree-12 filters for Source 1), capped-mass residual bounds, temperature transfer, coupled bootstrap rounds, Rayleigh–Ritz criterion, and `charpoly.roots` multiplicity extraction—is proved unconditionally in Lean 4. The *only* remaining premises of the two spin-1 theorems are the concrete finite physical thermal trace enclosures on Z_{spin1} ($n \leq 14$ for Candidate A; $n \leq 12$ for Source 1) and a single finite variational Rayleigh vector (ψ at $L = 112$ or $L = 72$).

2 General Purity-Bootstrap Library in Lean 4

Our core reusable framework comprises six modules:

- `PurityBootstrap/Basic.lean` and `OneD.lean`: Formalize 1D spatial purity $S(L,b) = Z(2L,b)/Z(L,b)^2$ and thermal (temporal) purity $T(L,b) = Z(L,2b)/Z(L,b)^2 = \sum_i p_i^2$ for any `SpectralSystem1D`, proving contraction, monotonicity, dyadic interpolation, and uniform spectral gap extraction: given *both* seed bounds $S(n_0,b_0) \geq 1 - u_0$ and $T(2n_0,b_0) \geq 1 - u_0$ together with the bootstrap

coefficient inequalities (parametrized by u_0 and C), every finite even ring $L \geq n_0$ has a unique ground state and uniform gap $\gamma_L > \frac{\log(1/(Cu_0))}{b_0}$. For Candidate A, $(n_0, b_0) = (896, 368)$ with $u_0 = 1/100$ and $C = 5$, yielding $\gamma_L > \frac{\log 20}{368}$.

- `PurityBootstrap/HigherDim.lean`: Formalizes Grover's general-d quadratic contraction and 2D ($2\times$) purity bootstrap [2].
- `PurityBootstrap/SymmetryTwists.lean`: Formalizes $\mathbb{Z}_2$ symmetry-twist multiplet splitting bounds (used in Candidate E) as well as Grover's 2D volume-law 4th-power step for future higher-dimensional applications.
- `PurityBootstrap/ExactCertificates.lean` and `MomentInit.lean`: Formalize exact upward-rounded rational bootstrap iterations together with the capped-mass lemma (`capped_mass_tsum`) and nonnegative polynomial-filter bound (`polynomial_filter_tsum`) linking raw moments to leading-eigenvalue enclosures.

3 Unconditional 2D Coupling-Rectangle Multiplet Theorem (Candidate E)

Consider the 1D interacting transverse-field Ising ring on $\ell = 2m$ qubits (Grover [2], Pauli normalization $J = 1$):

$$H_\ell(g,h) \;=\; -\sum_{i=0}^{\ell-1} Z_i Z_{i+1} \;-\; g\sum_{i=0}^{\ell-1} X_i X_{i+1} \;-\; h\sum_{i=0}^{\ell-1} X_i, \tag{1}$$

with global spin-flip symmetry $P = \bigotimes_{i=0}^{\ell-1} X_i$ partitioning the spectrum into $+1$ and -1 parity sectors.

Theorem 3.1 (Unconditional 2D Rectangle Multiplet Theorem)**.** *In `candidate_e_unconditional_full_hamiltonian_theorem`, for every coupling pair $(g,h) \in [\frac{15}{128}, \frac{16}{128}] \times [\frac{47}{128}, \frac{48}{128}]$ and every finite even ring length $\ell = 2m \geq 64$, let $E_0(\ell) \leq E_1(\ell) \leq \cdots \leq E_{2^\ell-1}(\ell)$ be all 2^ℓ eigenvalues of $H_\ell(g,h)$ (proved equal as a multiset to `H_full.charpoly.roots`). Then:*

1. *$E_0(\ell)$ is the unique ground energy of the $+1$ parity sector ($P = +1$) and $E_1(\ell)$ is the unique ground energy of the -1 parity sector ($P = -1$);*
2. *the low-energy doublet is exponentially narrow:*

$$0 \;\leq\; E_1(\ell) - E_0(\ell) \;\leq\; \frac{1}{20}\exp\!\left(-\frac{\log 20}{128}\,\ell\right);$$

3. *every higher excitation is uniformly gapped above the doublet:*

$$E_i(\ell) - E_1(\ell) \;>\; \frac{\log 8}{48} \;\approx\; 0.0433217 \qquad \forall\, 2 \leq i < 2^\ell.$$

Proof architecture inside Lean 4. Three ingredients make this 1D theorem 100% unconditional: (i) `LieTrotter.lean` and `IsingTrotter.lean` establish exact 3-way Lie–Trotter convergence and twisted checkerboard positive-semidefinite Gram transfer-matrix contractions for both untwisted and P-twisted partition functions, eliminating all abstract transfer-matrix hypotheses; (ii) `BlockGershgorin.lean` and `CandidateE/Corner*.lean` execute exact rational block-Gershgorin upper enclosures at all four corners of the coupling rectangle at $\ell = 8$, paired with an exact integer $\chi = 2$ periodic MPS Rayleigh witness at $\ell = 16$ (`IsingMPS.lean`) and log-convexity of sector traces across the rectangle (`MatrixTools.lean`, `CandidateE/RectangleTheorem.lean`), establishing $Q_+(8,6) \geq 877/1000$ uniformly across the rectangle; (iii) three exact upward-rounded 1D coupled purity-bootstrap rounds advance $(8,6) \to (64,48)$ and extract the doublet and gap bounds via `SymmetryTwists.lean`.

4 Eliminating All Abstract Transfer Hypotheses for Spin-1 Chains

For spin-1 chains, the purity bootstrap requires both an even-length `SpectralSystem1D` representation and *signed* sector-moment expansions holding across both even and odd ring lengths $n \geq 4$. Rather than postulating an infinite-dimensional compact transfer operator on L^2-path space, we build a completely elementary, finite-dimensional, kernel-verified transfer limit in four modules: `Spin1Hamiltonian.lean`, `Spin1Compactness.lean`, `Spin1Transfer.lean`, and `Spin1PhysicalGap.lean`.

4.1 Concrete Physical Spin-1 Hamiltonian and Weight-Basis Unitary Equivalence

In `Spin1Hamiltonian.lean`, we work in the real Cartesian spin-1 basis $\mathbb{R}^3$ with skew-symmetric generators $(G_\alpha)_{ij} = \varepsilon_{\alpha ij}$ ($\alpha \in \{0,1,2\}$) and diagonal anisotropy matrix SzSq_mat $= \mathrm{diag}(1,1,0)$. We construct an explicit 3×3 complex matrix $U_{\mathrm{cart}\to sz}$ and prove in Lean 4:

- `U_cart_to_sz_unitary`: $U_{\mathrm{cart}\to sz} U_{\mathrm{cart}\to sz}^\dagger = U_{\mathrm{cart}\to sz}^\dagger U_{\mathrm{cart}\to sz} = I_3$;
- `U_cart_to_sz_conj_S`: $U_{\mathrm{cart}\to sz}(-iG_\alpha)U_{\mathrm{cart}\to sz}^\dagger = S^\alpha$ for all $\alpha \in \{x,y,z\}$ in the standard weight basis $(+1,0,-1)$;
- `U_cart_to_sz_conj_SzSq`: $U_{\mathrm{cart}\to sz}\,\mathrm{SzSq_mat}\,U_{\mathrm{cart}\to sz}^\dagger = (S^z)^2 = \mathrm{diag}(1,0,1)$.

Because $(-i)^2 = -1$, each Heisenberg bond $\sum_\alpha S_j^\alpha \otimes (gS^\alpha g^{\mathsf{T}})_k$ equals $-\sum_\alpha (G_\alpha)_j \otimes (gG_\alpha g^{\mathsf{T}})_k$ in the real Cartesian basis, and tensoring $U_{\mathrm{cart}\to sz}$ across all n sites yields the many-body unitary equivalence to the textbook weight-basis ring Hamiltonian. For any $n \geq 3$, anisotropy $D \in \mathbb{R}$, and 3×3 real matrix g, the concrete $3^n \times 3^n$ real symmetric Hamiltonian is

$$H_{\mathrm{spin1}}(n,D,g) \;=\; \sum_{j=0}^{n-2} h_{j,j+1}(I) \;+\; h_{n-1,0}(g) \;+\; D\sum_{j=0}^{n-1}(\mathrm{SzSq_mat})_j, \tag{2}$$

where g rotates only site 0 of the wrap-around bond $(n-1,0)$. The concrete physical partition function with energy shift $a \in \mathbb{R}$ at inverse temperature $b > 0$ is

$$Z_{\mathrm{spin1}}(n,D,a,g,b) \;=\; \mathrm{Tr}\Big(\exp\big(-b\,(H_{\mathrm{spin1}}(n,D,g) + anI)\big)\Big),$$

proved equal to $\sum_{k\in(\mathrm{Fin}\,3)^n} e^{-b(E_k+an)}$ (`Z_spin1_eq_sum_exp`).

4.2 Finite-M No-Collision Equivariant Transfer Matrix and Trotter Limit

Fix $M \geq 1$ and $\tau = b/M$. On the finite configuration space $\mathrm{Word}(M) = (\mathrm{Fin}\,4)^M$ (where index 0 represents an unoccupied bond slice and indices $1,2,3$ carry the $\mathbb{R}^3$ vector label α), define the single-slice 3×3 matrix kernel

$$\Phi_\tau(u,v) \;=\; \begin{cases} I_3 & \text{if } u = 0, v = 0, \\ \sqrt{\tau}\,G_{v-1} & \text{if } u = 0, v \in \{1,2,3\}, \\ \sqrt{\tau}\,G_{u-1} & \text{if } u \in \{1,2,3\}, v = 0, \\ 0 & \text{if } u,v \in \{1,2,3\}, \end{cases}$$

and the $4^M \times 4^M$ real matrix $K_M(D,a,b)$ with entries

$$(K_M)_{w,w'} \;=\; \mathrm{Tr}_3\left(\prod_{t=0}^{M-1}\left(e^{-\tau V(D,a)}\,\Phi_\tau(w_t,w'_t)\right)\right), \qquad V(D,a) = D\,\mathrm{SzSq_mat} + aI_3.$$

We define the predicate `IsSpin1Rotg` requiring g to be orthogonal and covariant with the generators ($gG_\alpha g^{\mathsf{T}} = \sum_\beta g_{\beta\alpha}G_\beta$), and `IsAxialRotg` additionally requiring preservation of SzSq_mat. This zero-collision construction enjoys four exact properties proved in `Spin1Transfer.lean`:

1. **Exact real symmetry:** $K_M^{\mathsf{T}} = K_M$ (`K_mat_transpose`), because $\Phi_\tau(v,u) = \Phi_\tau(u,v)$ preserves the time-ordered product.

2. **Exact $O(2)/SO(3)$-equivariance:** With $U_M(g) = (1 \oplus g)^{\otimes M}$ on $\mathrm{Word}(M)$, we prove $U_M(g)K_M = K_M U_M(g)$ (`K_mat_comm_U`) for any g satisfying `IsSpin1Rotg` and preserving the onsite exponential $e^{-\tau V(D,a)}$—in particular, every axial rotation g satisfying `IsAxialRotg` (including $R_z(\pi/4)$, which pairs real irreducible sectors) at arbitrary D, and the cyclic axis permutation C_3 at $D = 0$.

3. **Exact cycle contraction and $O(1/M)$ Trotter limit:** For every $n \geq 3$ (even or odd), $M \geq 0$, and g satisfying `IsSpin1Rotg`, pure finite-sum expansion gives $\mathrm{Tr}(K_M^n U_M(g)) = \mathrm{Tr}(T_{\mathrm{slice}}(M)^M)$ (`trace_K_pow_mul_U_eq_trace_T_slice_pow`), where $T_{\mathrm{slice}}(M) = I - \frac{b}{M}(H_{\mathrm{spin1}} + anI) + O(1/M^2)$ in operator norm. By `tendsto_trace_pow_of_first_order` (`Spin1Compactness.lean`), for every g satisfying `IsSpin1Rotg`,

$$\lim_{M\to\infty} \mathrm{Tr}(K_M^n U_M(g)) \;=\; Z_{\mathrm{spin1}}(n, D, a, g, b) \qquad (\forall\, n \geq 3,\ b > 0).$$

4. **Uniform Hilbert–Schmidt bound:** Counting the $1+6 = 7$ nonzero single-slice label pairs and using $\|G_\alpha\|_\infty = 1$ yields the M-independent bound (`K_mat_hilbert_schmidt_le`)

$$\mathrm{Tr}(K_M^2) \;\leq\; 9\exp\big(2b(|D| + |a| + 3)\big) \;=:\; C(D, a, b).$$

4.3 Subsequential Signed-Moment Convergence and Real Symmetric Sector Projections

In `Spin1Compactness.lean`, `l2_seq_compact_moments` takes any family of finite real eigenvalue lists $\{f_{M,s,i}\}$ uniformly bounded in ℓ^2 ($\sum_i f_{M,s,i}^2 \leq C$), sorts each list by decreasing $|f_{M,s,i}|$ while preserving eigenvalue signs, pads with zeros, and applies Tychonoff sequential compactness on $[-\sqrt{C}, \sqrt{C}]^{\mathrm{Fin}\, k \times \mathbb{N}}$. The decreasing rearrangement gives $|f_{M,s,i}| \leq \sqrt{C/(i+1)}$, hence the uniform tail estimate

$$\sum_{i \geq N} |f_{M,s,i}|^n \;\leq\; C\left(\frac{C}{N+1}\right)^{(n-2)/2} \xrightarrow[N\to\infty]{} 0 \qquad (\forall\, n \geq 3),$$

establishing subsequential convergence of all signed n-th moments ($n \geq 3$, even and odd) with absolutely summable n-th powers for the limit sequences.

Working subspace-free on $P_s K_M P_s$ (which satisfies $\mathrm{Tr}((P_s K_M P_s)^n) = \mathrm{Tr}(P_s K_M^n)$ for $n \geq 1$ and $\mathrm{Tr}((P_s K_M P_s)^2) \leq \mathrm{Tr}(K_M^2)$), `Spin1Transfer.lean` proves exact real symmetric idempotent decompositions:

- $D_4 + R_z(\pi/4)$ **for arbitrary** D (`spin1_D4_sector_moments`)**:** Six real projections $A_1, B_1, B_2, A_2, E_+, E_-$ with orthogonal conjugacies $B_1 \cong B_2$ (via $U_M(R_z(\pi/4))$) and $E_+ \cong E_-$ (via $U_M(R_z(\pi/2))$), giving four real sequences $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ with absolutely summable n-th powers ($n \geq 4$) reproducing all four twisted traces $Z_{\mathrm{spin1}}(n, D, a, g, b)$ ($g \in \{I, R_z(\pi), R_x(\pi), R_z(\pi/2)\}$) via the exact character matrix in Table 2.

- $D_2 \rtimes C_3$ **for** $D = 0$ (`spin1_D2_sector_moments`)**:** Cyclic axis permutation $U_M(C_3)$ equates the three nontrivial D_2 sectors, yielding two real sequences λ_J, λ_N reproducing $Z_{\mathrm{spin1}}(n, 0, a, I, b) = J_n + 3N_n$ and $Z_{\mathrm{spin1}}(n, 0, a, R_x, b) = J_n - N_n$ for all $n \geq 4$.

- **Concrete** `SpectralSystem1D` (`spin1SpectralSystem`)**:** `spin1_spatial_eigs` supplies a real sequence with absolutely summable fourth powers reproducing $Z_{\mathrm{spin1}}(n, D, a, I, b)$ for all $n \geq 4$, instantiating `SpectralSystem1D` unconditionally for any $D, a \in \mathbb{R}$.

Seam g	A_1 (λ_0)	$B_1 \oplus B_2$ (λ_1)	A_2 (λ_2)	$E_+ \oplus E_-$ (λ_3)
I (`rotI`)	$+1$	$+2$	$+1$	$+2$
$R_z(\pi)$ (`rotZ`)	$+1$	$+2$	$+1$	-2
$R_x(\pi)$ (`rotX`)	$+1$	0	-1	0
$R_z(\pi/2)$ (`rotF`)	$+1$	-2	$+1$	0

Table 2: Exact real D_4 sector multiplicity matrix proved in `spin1_D4_sector_moments` and consumed by `CandidateA/Certificate.lean`.

5 Physical Spin-1 Haldane Gap Theorems: Candidate A and Source 1

Theorem 5.1 (Candidate A Physical Spin-1 Haldane Gap)**.** *In `candidate_a_physical_spin1_charpoly_gap_896`, let* $a = 1.242818 = 1242818/10^6$ *and* $\varepsilon_{\mathrm{CA}} = 10^{-5}$*. Suppose:*

1. *for every* $4 \le n \le 14$*, every seam* $g \in \{\texttt{rotI}, \texttt{rotZ}, \texttt{rotX}, \texttt{rotF}\}$*, and both* $b \in \{31/2, 23\}$*, the concrete physical trace* $Z_{\mathrm{spin1}}(n, 1/4, a, g, b)$ *lies within* $\varepsilon_{\mathrm{CA}}$ *of the corresponding rational table center in `CandidateA/Data.lean` (88 inequalities); and*
2. *there exists a nonzero real vector* $\psi \in \mathbb{R}^{\mathrm{Spin1Cfg}\,112}$ *satisfying the shifted Rayleigh inequality* $\langle \psi, H_{\mathrm{spin1}}(112, 1/4, I)\psi\rangle + 112a\|\psi\|_2^2 \le 0$*.*

Then for every finite even ring length $L \ge 896$*, the characteristic polynomial roots multiset of the concrete* $3^L \times 3^L$ *matrix* $H_{\mathrm{spin1}}(L, 1/4, I)$ *decomposes as* $E_0 ::_m \mathrm{rest}$ *with* $|\mathrm{rest}| + 1 = 3^L$ *and*

$$E - E_0 \;>\; \frac{\log 20}{368} \;\approx\; 0.00814058 \qquad (\forall\, E \in \mathrm{rest}).$$

Theorem 5.2 (Source 1 Physical Spin-1 Haldane Gap)**.** *In `source1_physical_spin1_charpoly_gap_2304`, let* $a_1 = 700741/500000$ *and* $\varepsilon_{\mathrm{S1}} = 30/10^6$*. Given the 18 concrete physical trace enclosures on* $Z_{\mathrm{spin1}}(n, 0, a_1, g, 49/4)$ *(*$4 \le n \le 12$*,* $g \in \{\texttt{rotI}, \texttt{rotX}\}$*) and a nonzero real vector* $\psi \in \mathbb{R}^{\mathrm{Spin1Cfg}\,72}$ *with* $\langle \psi, H_{\mathrm{spin1}}(72, 0, I)\psi\rangle + 72a_1\|\psi\|_2^2 \le 0$*, for every finite even* $L \ge 2304$ *the roots multiset of* $H_{\mathrm{spin1}}(L, 0, I)$ *decomposes as* $E_0 ::_m \mathrm{rest}$ *with* $|\mathrm{rest}| + 1 = 3^L$ *and* $E - E_0 > \frac{\log 20}{784}$ *for every* $E \in \mathrm{rest}$*.*

Inside Lean 4 vs. external numerical certification for Candidate A. Inside Lean 4 (`CandidateA/Data.lean`, `Certificate.lean`), all 16 degree-14 nonnegative polynomial filters are checked in the kernel via `decide+kernel`, combined with the exact 46/31 temperature-transfer temporal purity lemma ($31/2 \to 23$) and four upward-rounded coupled purity-bootstrap rounds $(56, 23) \to (896, 368)$ yielding the $(S(896, 368), T(1792, 368))$ seed pair with $u_0 = 1/100, C = 5$. The finite numerical inputs are certified externally by three standalone scripts:

- **7-site open-chain spectral lower bound (`scripts/certify_open7_shift.py`):** On the 7-site open window with onsite weight $3/14 = (6/7)(1/4)$ (so summing overlapping windows counts each bond 6 times and each onsite term 7 times), the script computes a floating-point Cholesky proposal of $A - 2I$ across S^z blocks $M = 0 \ldots 7$ (covering $M < 0$ by spin-flip symmetry, all 2,187 states), rounds it to integers, and certifies $A \succ I \succ 0$ via an exact overflow-guarded `int64` residual bound (worst $\|R\|_\infty = 1{,}216{,}644{,}908 < 2^{40}$). This yields $H_n(g) + \frac{21}{16}nI \succeq \frac{n}{336}I \succ 0$ for every $n \ge 7$ and every seam $g \in \{I, R_z, R_x, R_4\}$.
- **All 88 raw thermal trace enclosures (`scripts/verify_candidate_a_all88.py`):** Raw integer Chebyshev totals are produced using 128-bit integer recurrence arithmetic and 256-bit accumulation with zero floating-point operations and zero truncated states (`chebyshev_enclose.cpp`, `chebyshev_enclose_rx.cpp`, `chebyshev_enclose_cuda.cu`; bit-for-bit CPU/GPU

matched through $n = 11$, with 64/88 totals independently replayed on CPU, numerical full-matrix diagonalization cross-checks at $n = 4, 5, 6$, and rigorous direct Arb matrix-exponential checks at $n = 4$). The standalone stdlib-only exact rational verifier reads archived `W_int` totals and `CandidateA/Data.lean` centers, certifying all 88 rational intervals with worst-case endpoint distance $1.245 \times 10^{-6} < 10^{-5}$ (at $n = 6, I, b = 23$; $\leq 3.211 \times 10^{-8}$ at $n = 14$), leaving an 87.55% safety margin.

- $L = 112$ **variational trial witness (`scripts/check_trial.py`):** Encloses the periodic $\chi = 32$ two-site integer MPS Rayleigh quotient per site in 160-bit Arb arithmetic within $[-1.242820131121, -1.242820131120] < -1.242818$. Writing the Cartesian-basis complex vector as $u + iv$ for the real symmetric shifted matrix H_{shifted} gives $\langle u + iv, H_{\text{shifted}}(u + iv)\rangle = u^{\mathsf{T}} H_{\text{shifted}} u + v^{\mathsf{T}} H_{\text{shifted}} v < 0$, guaranteeing a nonzero real component satisfying Lean's real-vector Rayleigh premise.

6 Kernel Hygiene, Adversarial Auditing, and Reproducibility

The fail-closed audit script `scripts/check_axioms.lean` verifies that:

1. all 25 listed project modules are imported into the checked environment;
2. zero `.axiomInfo` declarations exist across all imported project modules (even unused); and
3. every one of the 1,509 project theorem constants (1,288 under `PurityBootstrap.*`) depends strictly on a subset of `[propext, Classical.choice, Quot.sound]`—which rules out `sorryAx` and `Lean.ofReduceBool` (`native_decide`) for every audited theorem.

In addition, the repository underwent five rounds of independent adversarial auditing using Codex (`gpt-6.2-astra`, high reasoning effort), including independent compilation of the CPU Chebyshev builders to confirm exact `rotF` seam orientation against H_{spin1} (max discrepancy $\leq 4.5 \times 10^{-15}$ vs. 2 under reversed twist $\texttt{rotF}^{\mathsf{T}}$), independent regeneration of 40/88 Candidate A totals and 14/18 Source 1 totals, replay of both $L = 112$ and $L = 72$ variational trials, individual `leanchecker` runs on all four new spin-1 modules, and verification of the source-file inventory and citations.

All checks reproduce cleanly from the repository root:

```
lake build
lake env lean scripts/check_axioms.lean
python3 -O scripts/certify_open7_shift.py
python3 -O scripts/verify_candidate_a_all88.py
python3 -O scripts/check_trial.py   # requires python-flint
```

7 Related Work and Outlook

Our formalization builds directly on the finite-temperature purity bootstrap of [1] and the symmetry-twist and higher-dimensional extensions of Grover [2]. Independently, Lu [3] formalized a finite-Gram/Trotter unconditional Lean 4 theorem for a mixed-field Ising chain and reported improved finite-input isotropic spin-1 gap bounds ($\gamma_L > 0.0729$ for even $L \geq 36$) together with a $D = 4$ single-ion anisotropy example. Earlier analytic stability of ground-state degeneracy and exponentially small splitting under weak symmetric perturbations was proved by Lapa and Levin [4], while Tasaki [5]—using the analytic machinery and numerical inputs of [1]—proved uniqueness among translation-invariant infinite-chain ground states for sufficiently small $|D|$, showing that this state is gapped and continuous in D, without an explicit finite-D certificate at $D = 1/4$.

Two natural next steps follow immediately from the infrastructure in `haldane-purity-lean`: (i) combining `HigherDim.lean` with a 2D checkerboard Gram–Trotter construction to establish an unconditional 2D quantum spin-model multiplet and spectral gap theorem on even tori inside

Lean 4; and (ii) sweeping additional single-ion anisotropies D toward the Haldane-to-large-D phase boundary ($D_c \approx 0.97$) using our exact CUDA Chebyshev and Lean 4 pipeline.

References

[1] OpenAI, *The periodic spin-one Haldane gap*, preprint, September 24, 2026.

[2] T. Grover, *Extensions of the Haldane gap proof: higher dimensions, near degeneracy, fermions and tensor-network states*, `arXiv:2611.07400`, 2026.

[3] S. Lu, *Bridging Gaps with Tensor Networks: From the Haldane Gap and Beyond*, `arXiv:2611.11049`, 2026.

[4] M. F. Lapa and M. Levin, *Stability of ground state degeneracy to long-range interactions*, `arXiv:2107.11396`, 2021.

[5] H. Tasaki, *On the uniqueness of translation-invariant ground states of the $S = 1$ Heisenberg and J–D antiferromagnetic chains*, `arXiv:2610.34330v2`, 2026.